Onset of Convection in a Porous Medium with Internal Heat Generation and Downward Flow

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The criteria for the onset of natural convection in a fluid-saturated porous medium with a distributed heat source from absorbed solar radiation and an externally imposed downward flow of the fluid are determined. The Brinkman-extended Darcy formulation is used in the equations of motion. Linear stability analysis is performed on the continuity, momentum, and energy equations and the resulting linearized equations are solved using a Galerkin method. The stability criteria so obtained are given in terms of the internal and external Rayleigh numbers, porous medium permeability and extinction coefficient, and Peclet number of the imposed downward flow. The predicted stability criteria in the limiting cases of no downward flow and no heat generation are in agreement with those obtained by others. These results can be applied to the evaluation of a possible improved solar pond design, an end that motivated this study.

Nomenclature

A_m = constant in the sine series solution
a = resultant horizontal wave number, $a^2 = n_1^2 + n_2^2$
B_m = constant in Eq. (29)
C_m = constant in Eq. (31)
C_{ki} = coefficients of the determinant in Eq. (34)
C_p = specific heat at constant pressure, J/kg-K
D_m = constant in Eq. (31)
E_m = constant in Eq. (31)
F_m = constant in Eq. (31)
g = acceleration of gravity, m/s^2
I_i^{nm} = ith integral in Eq. (33)
K = permeability of porous medium, m^2
k = thermal conductivity, W/m-K
L = layer depth, m
n_1,n_2 = horizontal wave numbers in the x and y directions, respectively
N = dimensionless extinction coefficient, $N = \eta L$
Pe = Peclet number, $Pe = w_o L/\alpha_m$
p = pressure, Pa
p_o = steady-state pressure, Pa
p' = pressure disturbance, Pa
Q_v = steady volumetric heat source, W/m^3
q_o = heat flux at the surface of the layer, W/m^2
q_r = heat flux at any depth z, W/m^2
Ra_E = external Rayleigh number, $Ra_E = g\beta\, \Delta T L^3/\alpha_m v_f$
Ra'_E = modified external Rayleigh number, $Ra'_E = g\beta\, \Delta TKL/\alpha_m v_f$
Ra_I = internal Rayleigh number, $Ra_I = g\beta q_o L^4/k_m \alpha_m v_f$
Ra'_I = modified internal Rayleigh number, $Ra'_I = g\beta q_o KL^2/k_m \alpha_m v_f$
R^* = ratio of internal to external Rayleigh number, $R^* = Ra_I/Ra_E$
T = temperature, K
ΔT = temperature difference between upper and lower surface, $\Delta T = T_b - T_a$, K
t = time, s
u = velocity disturbance in the x direction, m/s
$\bar{V}$ = velocity vector
v = velocity disturbance in the y direction, m/s
W = dimensionless velocity disturbance in the z direction, $W = wL/\alpha_m$
w = velocity disturbance in the z direction, m/s
w_o = magnitude of the imposed downward velocity, m/s
X = dimensionless distance in the x direction, $X = x/L$
x = Cartesian coordinate, m
Y = dimensionless distance in the y direction, $Y = y/L$
y = Cartesian coordinate, m
Z = dimensionless distance in the z direction, $Z = z/L$
z = Cartesian coordinate, m

Greek symbols

α_m = effective thermal diffusivity, $\alpha_m = k_m/(\rho C_p)_f$
β = thermal expansion coefficient, K^{-1}
β_1 = constant in Eq. (30)
β_2 = constant in Eq. (30)
ε = porosity
η = extinction coefficient, m^{-1}
θ = temperature disturbance, K
Θ = dimensionless temperature disturbance, $\Theta = g\beta\theta L^3/\alpha_m v_f$
μ = dynamic viscosity, kg/m-s
v = kinematic viscosity, m^2/s
ρ = fluid density, kg/m^3
ρ_c = density at the reference state where the temperature $T = 0$, kg/m^3
∇ = del operator, $\nabla = \hat{i}(\partial/\partial x) + \hat{j}(\partial/\partial y) + \hat{k}(\partial/\partial z)$
∇^2 = Laplacian operator, $\nabla^2 = (\partial^2/\partial x^2) + (\partial^2/\partial y^2) + (\partial^2/\partial z^2)$, $\nabla_1^2 = (\partial^2/\partial x^2) + (\partial^2/\partial y^2)$

Superscripts

— = vector quantity
n = incremental integer
m = incremental integer

Subscripts

a = upper surface
b = bottom surface
f = fluid properties
m = fluid-saturated porous medium properties
s = properties of the solid used as the porous medium
o = equilibrium or steady-state condition

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Introduction

THIS study of natural convection in a fluid-saturated porous medium with a distributed heat source and an externally imposed downward flow of the fluid was motivated by an investigation of a saltless solar pond design. The main feature of the design consists of continuously withdrawing warm water from the bottom of the pond for distribution to the point of use, while cool make-up water is continuously added at the same rate at the top. This creates a gentle flow downward in the direction of increasing temperature. On physical grounds this downward-convecting pond is potentially superior to the salt-stratified stagnant solar pond because it eliminates the difficulties of using salt, and it can be more efficient since downward convection of stored energy in the water toward the bottom opposes conduction through the water toward the top.

A preliminary numerical study by Smiley and Burmeister[1] compared the solar collection efficiencies of the stagnant salt-gradient and the downward-convecting solar ponds, indicating that the amount of useful heat collected with the downward-convecting pond could be nearly 35% greater. That study, however, suggested that the downward convective velocity must be small, of the order of 10^{-3} m/h, to achieve usefully high bottom temperatures. Since such a small downward convective velocity is unable to overcome upward natural convection by itself, a porous medium was introduced into the solar pond design. The stability criteria pertinent to this physical situation are needed in order to proceed with the evaluation of the envisioned solar pond scheme. In the present study, a mathematical model of a horizontal fluid-saturated porous medium with externally imposed downward convection and distributed internal heat generation is numerically solved to determine these needed stability criteria.

Several studies have been done on the onset of convection in a fluid-saturated porous medium. Lapwood[2] was one of the early investigators to determine the onset of convection in a fluid-saturated porous medium that is heated from below. Subsequently, several investigators, including Holst and Aziz,[3] Gupta and Joseph,[4] Rudraiah et al.,[5] Facas and Farouk,[6] and Prasad and Kulacki,[7] studied the phenomenon of natural convection in a fluid-saturated porous medium for horizontal layers and vertical cavities and obtained steady-state and transient numerical solutions for several boundary conditions. Experimental results due to Katto and Masuoka,[8] Holst and Aziz,[3] and Castinel and Combarnous[9] are also available for a variety of configurations and several types of porous media. The stability of the fluid-saturated porous medium when heated internally within the layer was considered by Gasser and Kazimi,[10] Tveitereid,[11] Kaviany,[12] and Somerton et al.[13] Among them, Gasser and Kazimi[10] used the method of linear stability of small disturbances to solve the problem. They defined an internal Rayleigh number and an external Rayleigh number and predicted the critical values of the two numbers below which natural convection does not occur. Kaviany[12] used the linear stability theory to solve the same problem but with upper and lower free surfaces. He also used the amplification theory to examine the effects of porosity, permeability, and Prandtl number on the time of the onset of convection. Burretta[14] and Sun[15] experimentally determined the critical internal Rayleigh number of an internally heated porous medium for the case where the layer is bounded by an isothermal rigid upper surface and an adiabatic rigid lower surface.

Problem Formulation

The method used in this study borrows heavily from the approach of Gasser and Kazimi,[10] but accounts for the externally imposed downward velocity and the distributed internal heat source created by the absorbed solar flux.

In order to formulate a problem that is amenable to solution, several simplifying assumptions were made. In general the lateral dimensions of the pond are extremely large compared to the depth; consequently, the thermal flow through the surrounding walls is negligible compared to the vertical heat flow, and a one-dimensional system is used. The temperature of the make-up water added at the upper surface is assumed to be equal to the temperature of the environmental air. This assumption seems to be reasonable due to the fact that the cooling caused by evaporation at the surface is approximately counteracted by the absorption of the long wavelength portion of the solar radiation in a thin layer at the surface as explained by Weinberger.[16] The porous medium is assumed to be homogeneous and radiant energy absorbed at a distance below the top of the pond is considered to be known, thus neglecting reflections within the pond. As the fluid permeates the porous medium, the gross motion of the fluid is taken to be at a constant downward velocity. The Boussinesq approximation is used, disregarding any variation of density with temperature except in the body force term in the equation of motion.

The physical properties of the fluid and porous medium, which include viscosity, thermal conductivity, specific heat, thermal expansion coefficient, and permeability, are also assumed to be constant. Under these assumptions, and for the situation shown in Fig. 1, the describing equations are the continuity equation

$$\frac{D'\rho}{Dt} + \rho\nabla\cdot\bar{V} = 0 \tag{1}$$

in which $D'/Dt = \varepsilon\,\partial/\partial t + \bar{V}\cdot\nabla$, the equation of motion

$$\rho\frac{D''\bar{V}}{Dt} = (0,0,-\rho g) - \nabla p - \frac{\mu}{K}\bar{V} + \mu\nabla^2\bar{V} \tag{2}$$

in which $D''/Dt = \varepsilon^{-1}\,\partial/\partial t + \varepsilon^{-2}\,\bar{V}\cdot\nabla$, and the energy equation

$$\rho C_p\frac{D'''T}{Dt} = k_m\nabla^2 T + Q_v \tag{3}$$

in which $D'''/Dt = [(\rho C_p)_m/\rho C_p]\,\partial/\partial t + \bar{V}\cdot\nabla$.

Darcy's law in the form

$$\nabla p_m = -\frac{\mu}{K}\bar{V} \tag{4}$$

is used to modify the equation of motion [Eq. (2)] and account for the pressure gradient due to flow resistance in the porous medium. Also, in Eq. (2) the Brinkman term $\mu\nabla^2\bar{V}$ is included to more fully account for viscous contributions which are significant at high Rayleigh and Darcy numbers, as presented by Tong and Subramanian[17] for a boundary layer analysis. The variation of density with temperature in the body force term of Eq. (2) is expressed by

$$\rho = \rho_c(1-\beta T) \tag{5}$$

where ρ_c is the density at the reference state where the temperature $T = 0$.

A well-established procedure for this type of problem, developed by Jeffreys,[18] considers small disturbances, or pertuba-

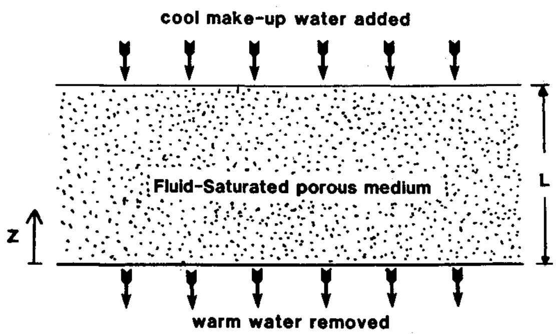

Fig. 1 Physical configuration of the problem.

tions, from the equilibrium condition to occur when the system becomes unstable. An instantaneous quantity is taken to be the sum of its steady-state value and a perturbation as

$$T = T_o + \theta \tag{6}$$

$$p = p_o + p' \tag{7}$$

and

$$\bar{V} = u\hat{i} + v\hat{j} + (w - w_o)\hat{k} \tag{8}$$

where T_o, p_o, and w_o are the temperature, pressure, and velocity, respectively, at the equilibrium state and θ, p', and (u,v,w) are disturbances, or perturbations, from the equilibrium state.

In the absence of natural convection at the equilibrium state, the equations describing steady-state conditions are found from Eqs. (1–3) to be

$$\bar{V}_o = -w_o\hat{k} \tag{9}$$

$$0 = -\rho_o g - \frac{\mathrm{d}p_o}{\mathrm{d}z} + \frac{\mu}{K} w_o \tag{10}$$

and

$$\frac{\mathrm{d}^2 T_o}{\mathrm{d}z^2} + \frac{w_o}{\alpha_m}\frac{\mathrm{d}T_o}{\mathrm{d}z} + \frac{1}{k_m}\frac{\mathrm{d}q_r}{\mathrm{d}z} = 0 \tag{11}$$

Here q_r represents the amount of radiation that has penetrated to a depth $L - z$ and, neglecting scattering and reflection from the bottom of the pond as discussed by Daniel et al.,[19] is given by Lambert's law in the form

$$q_r = q_o \exp[-\eta(L - z)] \tag{12}$$

The dimensionless steady-state temperature distribution is found from Eqs. (11) and (12) and from the appropriate boundary conditions to be

$$T_o^*(Z) = -\frac{R^*}{N + Pe}\exp[-N(1 - Z)] + C^*[\exp(-PeZ) - \exp(-Pe)] + \frac{R^*}{N + Pe} \tag{13a}$$

where

$$C^* = \frac{1}{1 - \exp(-Pe)} - \frac{R^*}{N + Pe}\,\frac{1 - \exp(-N)}{1 - \exp(-Pe)} \tag{13b}$$

and

$$R^* = \frac{Ra_I}{Ra_E} \tag{13c}$$

The dimensionless steady-state temperature distribution is shown in Fig. 2 for $R^* = 10$ and various values of the dimensionless extinction coefficient N and Peclet number Pe, based on the externally imposed downward velocity w_o.

Use of the steady-state relationships of Eqs. (9) and (10) and the dimensional forms of Eqs. (13a–13c) in the conservation equations of Eqs. (1–3) enables the perturbation form of Eqs. (1–3) to be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{14}$$

for the continuity equation

$$\rho\frac{D}{Dt}(u,v,w) = (0,0,\rho g\beta\theta) - \left(\frac{\partial p'}{\partial x}, \frac{\partial p'}{\partial y}, \frac{\partial p'}{\partial z}\right) - \frac{\mu}{K}(u,v,w) + \mu\nabla^2(u,v,w) \tag{15}$$

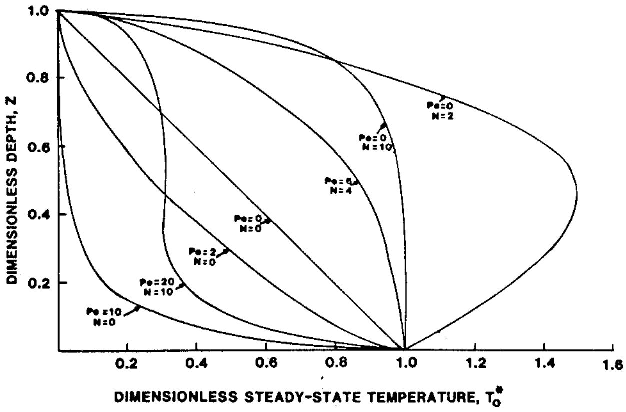

Fig. 2 Steady-state temperature distribution.

for the equation of motion, and

$$(\rho C_p)_m\frac{\partial\theta}{\partial t} + (\rho C_p)_f\left[u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y} + (w - w_o)\frac{\partial(T_o + \theta)}{\partial z}\right] = k_m\left[\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2(T_o + \theta)}{\partial z^2}\right] + Q_v \tag{16}$$

for the energy equation.

Jeffreys[18] showed that the onset of convection occurs when the departure from the steady-state is different from zero and independent of time. That is, the condition of marginal stability is given by $\partial/\partial t = 0$. Additional simplifications are then available by making use of the condition of marginal stability and the fact that the departures from the equilibrium condition are small enough that second-order terms can be neglected. Accordingly, first-order linear equations are formulated from Eqs. (15) and (16) to be

$$0 = (0,0,\rho g\beta\theta) - \left(\frac{\partial p'}{\partial x}, \frac{\partial p'}{\partial y}, \frac{\partial p'}{\partial z}\right) - \frac{\mu}{K}(u,v,w) + \mu\nabla^2(u,v,w) \tag{17}$$

from the equation of motion and

$$(\rho C_p)_f\left(w\frac{\partial T_o}{\partial z} - w_o\frac{\partial T_o}{\partial z} - w_o\frac{\partial\theta}{\partial z}\right) = k_m\left(\nabla^2\theta + \frac{\partial T_o}{\partial z^2}\right) + Q_v \tag{18}$$

from the energy equation. The convective term $\rho w_o\, \partial w/\partial z$ is neglected on the left-hand side of Eq. (17), because the effects of the porous medium are much larger at the low permeabilities and low velocities envisioned; the convective term could be important when permeability is larger, of course. Equation (18) is converted into dimensionless form and combined with Eq. (13) to obtain

$$\left\{-\frac{NRa_I}{N + Pe}\exp[-N(1 - Z)] - \left(Ra_E - \frac{1 - \exp(-N)}{N + Pe}Ra_I\right) \times \frac{Pe}{1 - \exp(-Pe)}\exp(-PeZ)\right\}W = \nabla^2\Theta + Pe\frac{\partial\Theta}{\partial Z} \tag{19}$$

where Ra_I and Ra_E are the internal and external Rayleigh numbers, respectively, and Pe is the Peclet number based on the externally imposed downward velocity w_o.

Next, the divergence of the equation of motion [Eq. (17)] is combined with the result of applying the ∇^2 operator to the Z-direction equation of motion [Eq. (17)] to eliminate pressure and obtain, with the aid of the continuity equation [Eq. (14)], another equation relating the dimensionless temperature disturbance θ to the dimensionless velocity disturbance. The result is

$$\left(\nabla^2 - \frac{L^2}{K}\right)\nabla^2 W = -\nabla_1^2 \Theta \tag{20}$$

where the del operators are now in dimensionless form. Equations (19) and (20) can now be solved simultaneously. The solutions are assumed to be of the forms

$$\Theta(X,Y,Z) = \Theta(Z)\sin(n_1 X)\sin(n_2 Y) \tag{21}$$

and

$$W(X,Y,Z) = W(Z)\sin(n_1 X)\sin(n_2 Y) \tag{22}$$

where n_1 and n_2 are integers. Equations (21) and (22) are substituted into Eqs. (19) and (20) to obtain

$$\left\{-\frac{NRa_I}{N+Pe}\exp[-N(1-Z)] - \left[Ra_E - \frac{1-\exp(-N)}{N+Pe}Ra_I\right] \times \frac{Pe\exp(-PeZ)}{1-\exp(-Pe)}\right\} W = (D^2-a^2)\,\Theta(Z) + Pe\frac{\mathrm{d}}{\mathrm{d}Z}[\Theta(Z)] \tag{23}$$

and

$$\frac{\mathrm{d}^4 W}{\mathrm{d}Z^4} - \left(2a^2 + \frac{L^2}{K}\right)\frac{\mathrm{d}^2 W}{\mathrm{d}Z^2} + a^2\left(a^2 + \frac{L^2}{K}\right) W = a^2\Theta(Z) \tag{24}$$

with

$$a^2 = n_1^2 + n_2^2 \tag{25}$$

where a is called the wave number.

Equations (23) and (24) can be combined to yield a single, sixth-order, ordinary differential equation similar to the final form of the perturbation equation derived by Sparrow et al.[20] or the equation derived more recently by Kaviany.[12] The complexity of the perturbation equations in this analysis stems mainly from the variable source term of Eq. (12) and the externally imposed downward flow modeled through Eq. (13).

The simultaneous solution of Eqs. (23) and (24) must satisfy the conditions of a free surface at the upper boundary and a rigid surface at the lower boundary. For the solar pond application, the rigid boundary condition at the bottom is used to simulate a porous rigid surface which is needed to hold up the porous medium while allowing a uniform downward flow. The top surface of the solar pond is exposed to the atmosphere. Mathematically, the boundary conditions are

$$W = 0 = \frac{\mathrm{d}W}{\mathrm{d}Z} \quad \text{at } Z = 0 \tag{26a}$$

$$W = 0 = \frac{\mathrm{d}^2 W}{\mathrm{d}Z^2} \quad \text{at } Z = 1 \tag{26b}$$

$$\Theta = 0 \quad \text{at } Z = 0 \tag{26c}$$

$$\Theta = 0 \quad \text{at } Z = 1 \tag{26d}$$

Problem Solution

The conditions for the onset of convection are determined by simultaneously solving Eqs. (23), (24), and (26). For the temperature disturbance, a Fourier sine series solution is assumed of the form

$$\Theta(Z) = \frac{1}{a^2}\sum_m A_m \sin(m\pi Z) \tag{27}$$

This form of the solution satisfies the boundary conditions in Eqs. (26c) and (26d). Substituting Eq. (27) into Eq. (24) yields

$$\frac{\mathrm{d}^4 W}{\mathrm{d}Z^4} - \left(2a^2 + \frac{L^2}{K}\right)\frac{\mathrm{d}^2 W}{\mathrm{d}Z^2} + a^2\left(a^2 + \frac{L^2}{K}\right) W = \sum_m A_m \sin(m\pi Z) \tag{28}$$

The particular solution W_p for Eq. (28) is of the form

$$W_p = \sum_m B_m \sin(m\pi Z) \tag{29a}$$

where B_m is given by

$$B_m = \frac{A_m}{(m\pi)^4 + \left(2a^2 + \frac{L^2}{K}\right)(m\pi)^2 + a^2\left(a^2 + \frac{L^2}{K}\right)} \tag{29b}$$

The homogeneous solution W_h of Eq. (28) is of the form

$$W_h = C\sinh(\beta_1 Z) + D\sinh(\beta_2 Z) + E\cosh(\beta_1 Z) + F\cosh(\beta_2 Z) \tag{30a}$$

where

$$\beta_1^2 = a^2 \tag{30b}$$

and

$$\beta_2^2 = a^2 + \frac{L^2}{K} \tag{30c}$$

Combining Eqs. (29) and (30) gives the general solution as

$$W = \sum_m B_m[\sin(m\pi Z) + C_m\sinh(\beta_1 Z) + D_m\sinh(\beta_2 Z) + E_m\cosh(\beta_1 Z) + F_m\cosh(\beta_2 Z)] \tag{31}$$

The constants C_m through F_m are evaluated by application of the boundary conditions of Eqs. (26a) and (26b) (see Appendix).

To complete the mathematical solution, Eq. (31) is substituted into Eq. (23) to obtain

$$\sum_m A_m\left[-\frac{(m\pi)^2 + a^2}{a^2}\right]\sin(m\pi Z) + Pe\sum_m A_m\left(\frac{m\pi}{a^2}\right)\cos(m\pi Z) = \left\{-\frac{NRa_I}{N+Pe}\exp[-N(1-Z)] - \left(Ra_E - \frac{1-\exp(-N)}{N+Pe}Ra_I\right)\frac{Pe\exp(-PeZ)}{1-\exp(-Pe)}\right\} \times \sum_m B_m[\sin(m\pi Z) + C_m\sinh(\beta_1 Z) + D_m\sinh(\beta_2 Z) + E_m\cosh(\beta_1 Z) + F_m\cosh(\beta_2 Z)] \tag{32}$$

Following a standard procedure with Fourier series for evaluating the coefficients, Eq. (32) is multiplied by $\sin(n\pi Z)$ and integrated between $Z = 0$ and $Z = 1$ to obtain the set of n equations

$$\sum_m \left\{ \frac{(m\pi)^2 + a^2}{a^2} I_{10}^{nm} - Pe\left(\frac{m\pi}{a^2}\right) I_{11}^{nm} - \frac{1}{(m\pi)^4 + \left(2a^2 + \frac{L^2}{K}\right)(m\pi)^2 + a^2\left(a^2 + \frac{L^2}{K}\right)} \times \left[\left(Ra_E - \frac{1-\exp(-N)}{N+Pe} Ra_I\right) \times \frac{Pe}{1-\exp(-Pe)} (I_0^{nm} + C_m I_1^n + D_m I_2^n + E_m I_3^n + F_m I_4^n) + \frac{N\exp(-N)}{N+Pe} Ra_I (I_5^{nm} + C_m I_6^n + D_m I_7^n + E_m I_8^n + F_m I_9^n)\right]\right\} A_m = 0 \qquad (33)$$

The integrals I_0 through I_{11} are given in the Appendix. In order for the solution of Eq. (33) to be nontrivial (i.e., $A_m \neq 0$), the $n \times m$ determinant obtained from the matrix of the coefficients of A_m must vanish with $n = m$. Accordingly,

$$\begin{vmatrix} C_{11} & C_{12} & \cdots & C_{1i} & \cdots & C_{1m} \\ C_{21} & C_{22} & \cdots & C_{2i} & \cdots & C_{2m} \\ \vdots & & & & & \\ C_{k1} & C_{k2} & \cdots & C_{ki} & \cdots & C_{km} \\ \vdots & & & & & \\ C_{n1} & C_{n2} & \cdots & C_{ni} & \cdots & C_{nm} \end{vmatrix}_{\det} = 0 \qquad (34)$$

The coefficients C_{ki}, the quantities in braces in Eq. (33), are functions of the external Rayleigh number Ra_E, the internal Rayleigh number Ra_I, the Peclet number Pe, the dimensionless porous medium permeability K/L^2, the dimensionless extinction coefficient N, and the wave number a. Therefore, solution of Eq. (34) generates a relationship between these dimensionless parameters.

Muller's iterative numerical method described by Kristiansen[21] was implemented with a computer program to obtain the stability criteria by finding the Rayleigh number (say, Ra_I) that causes the determinant of Eq. (34) to vanish when the alternate Rayleigh number (say, Ra_E) and the wave number are specified. For fixed values of the extinction coefficient, porous medium permeability, and Reynolds number of the downward flow, a relationship was generated between the internal Rayleigh number and the external Rayleigh number which then provided the stability criterion.

In order to determine the relationship between the external Rayleigh number and the internal Rayleigh number, calculations were made for several values of the wave number a; after specifying one of the Rayleigh numbers the other was determined by solving Eq. (34). Solution of a 5×5 determinant, the results of several trials, revealed that a fifth-order determinant in Eq. (34) provided sufficient accuracy, yielding five roots. A graph of one of the Rayleigh numbers vs the wave number showed five curves called harmonics. The critical Rayleigh number at which natural convection begins is taken to be the minimum of the lowest harmonic.

Results and Discussion

Calculations were first made for cases studied by previous investigators. Lapwood's[2] criterion for the case of no internal heat generation ($Ra_I = 0$) and no externally imposed downward flow ($Pe = 0$) is shown in Fig. 3 to agree with the results of the present study.

The case of a uniformly distributed internal heat source without downward flow ($Pe = 0$) was investigated by Gasser and Kazimi.[10] They found that if both Rayleigh numbers are multiplied by the dimensionless permeability, the resulting critical modified Rayleigh numbers fall on a single curve

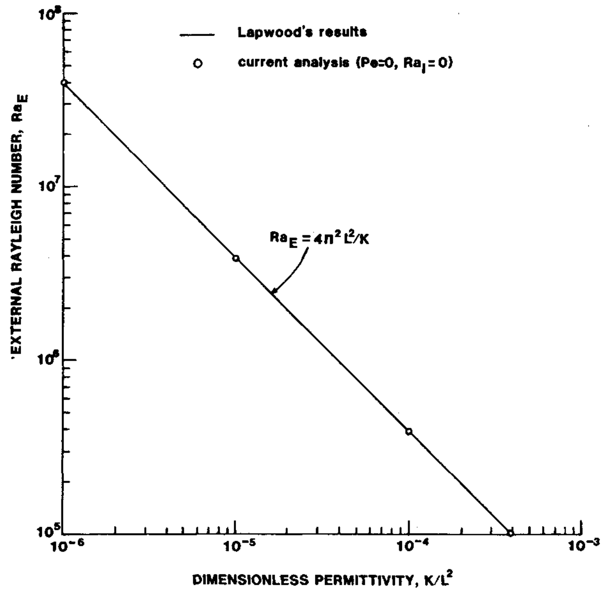

Fig. 3 Comparison between present results and Lapwood's results for $Ra_I = 0$ and $Pe = 0$.

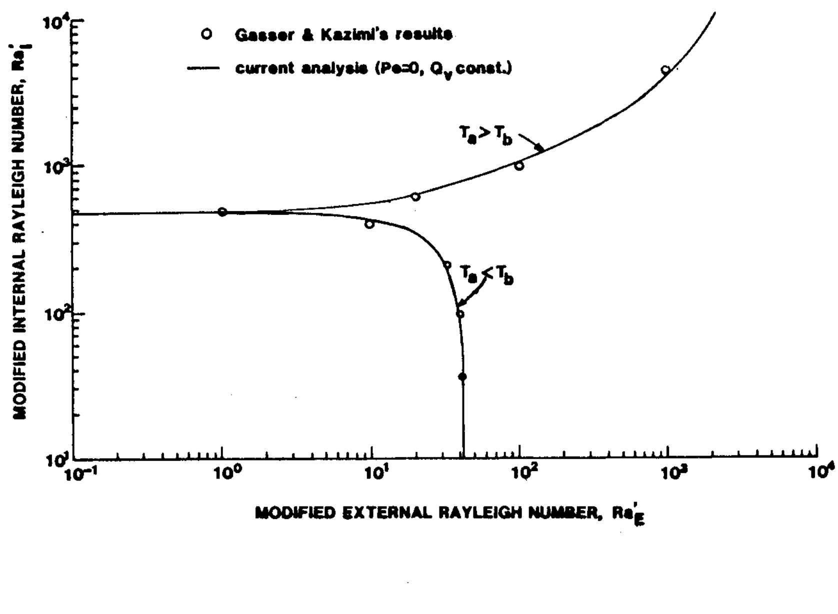

Fig. 4 Stability curves for the case of constant internal heat source; comparison with Gasser and Kazimi's results.

that represents the stability criterion. These modified Rayleigh numbers are defined as

$$Ra'_E = \frac{g\beta \Delta T K L}{\alpha_m \nu_f} \tag{35}$$

for the modified external Rayleigh number and

$$Ra'_I = \frac{g\beta q_o K L^2}{k_m \alpha_m \nu_f} \tag{36}$$

for the modified internal Rayleigh number. The results of the present study are shown in Fig. 4 to agree with those of Gasser and Kazimi for both destabilizing ($T_b > T_a$) and stabilizing ($T_b < T_a$) temperatures at the boundaries.

Next, calculations for the previously unstudied case of an externally imposed downward flow and no absorbed solar radiation (i.e., no internal heat generation, $Ra_I = 0$) were performed. The critical external Rayleigh number depends upon the Peclet number of the externally imposed downward flow, as shown in Fig. 5, for several values of the dimensionless permeability. As expected, the critical external Rayleigh number increases either with increasing downward velocity or with decreasing porous medium permeability.

To determine the dependence of the internal Rayleigh number upon the Peclet number and the extinction coefficient, calculations were then made for the case of no temperature difference between the top and bottom surfaces ($Ra_E = 0$). The results are shown in Fig. 6, where it is seen that the internal Rayleigh number increases with increasing extinction coefficient and Peclet number. Preliminary experiments without downward convection were performed by Hadim[22] to establish the range of extinction coefficients likely to be of interest. Additional results for the same case ($Ra_E = 0$) for the various values of the dimensionless permeability are shown in Fig. 7.

Finally, calculations were made to determine the stability criteria which were displayed in the form of a graph to show the relationship between the modified internal Rayleigh number and the modified external Rayleigh number. A sample stability curve for the case $Pe = 29.1$ and $N = 1$ is shown in Fig.

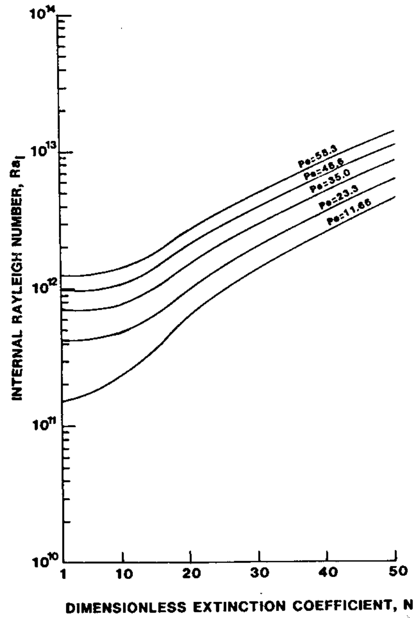

Fig. 6 Effects of the extinction coefficient on the internal Rayleigh number.

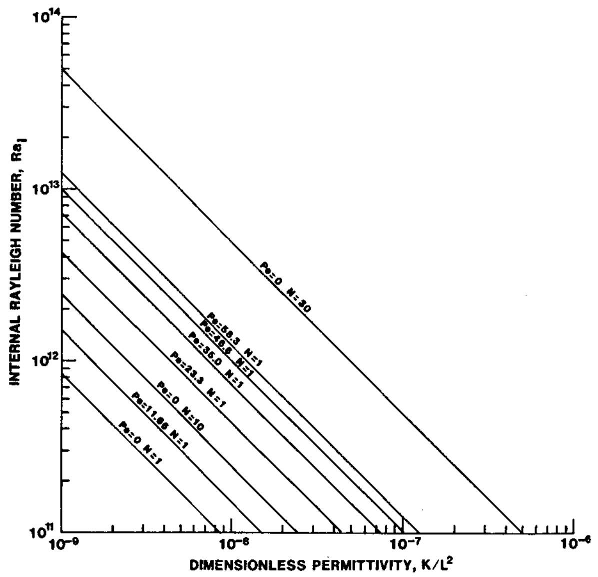

Fig. 7 Effects of permeability on the internal Rayleigh number.

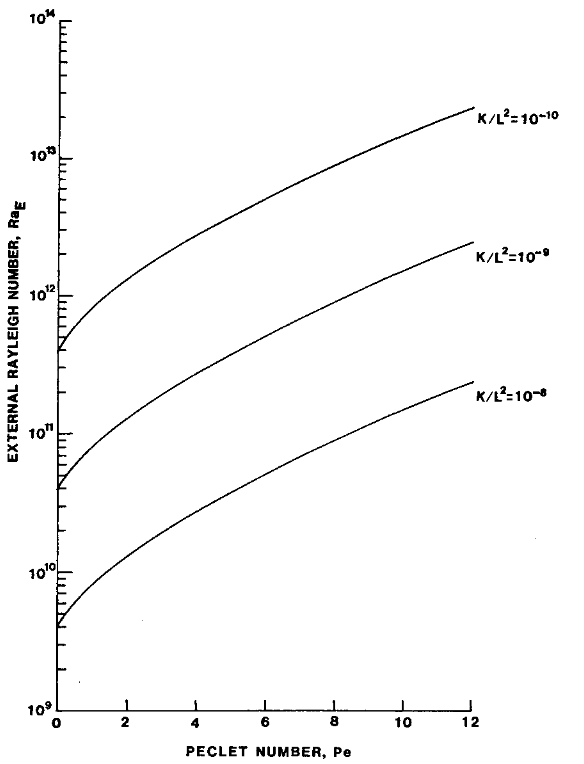

Fig. 5 Effects of downward velocity on the external Rayleigh number.

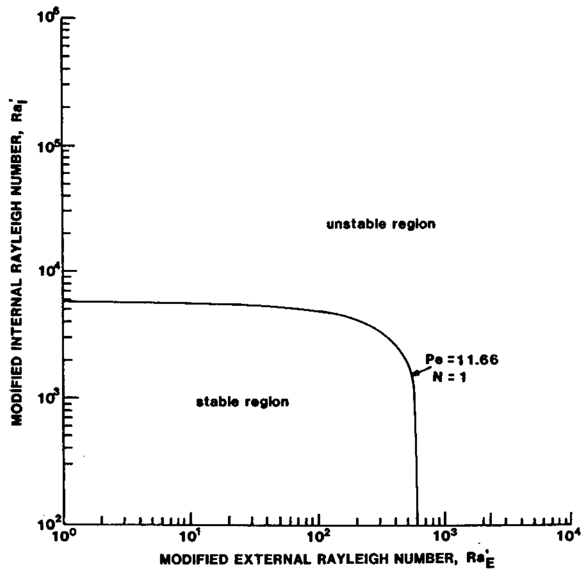

Fig. 8 Sample stability curve.

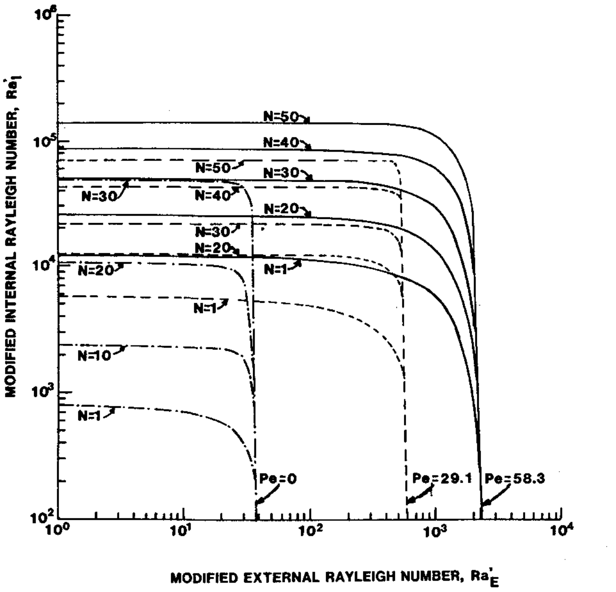

Fig. 9 Stability curves for several values of the Peclet number and extinction coefficient.

$$\begin{vmatrix} 0 & 0 & 1 & 1 \\ \beta_1 & \beta_2 & 0 & 0 \\ \sinh(\beta_1) & \sinh(\beta_2) & \cosh(\beta_1) & \cosh(\beta_2) \\ \beta_1^2 \sinh(\beta_1) & \beta_2^2 \sinh(\beta_2) & \beta_1^2 \cosh(\beta_1) & \beta_2^2 \cosh(\beta_2) \end{vmatrix} \begin{vmatrix} C_m \\ D_m \\ E_m \\ F_m \end{vmatrix} = \begin{vmatrix} 0 \\ -m\pi \\ 0 \\ 0 \end{vmatrix} \quad \text{(A3)}$$

8 to separate the stable and unstable regions. The more complete stability curves shown in Fig. 9 were generated for several sets of values of the Peclet number of the downward flow and the extinction coefficient.

The results of the present study can be applied to the evaluation of a possible solar pond design, the saltless solar pond described earlier. Details of such an application, as well as the results of preliminary measurements of permeability and extinction coefficient of water-saturated fiberglass, are presented elsewhere.[22] As described by Hadim,[22] the results from Fig. 9 are combined with results from an analysis that determines the theoretical maximum bottom temperature that can be obtained in a downward convecting solar pond filled with a porous medium. Preliminary design calculations were performed to study the feasibility of such a pond. The results suggest that a 20°C temperature difference might be obtained in a solar pond of 0.3 m depth with a 0.013 m/h downward flow and a porous medium of 0.141×10^{-9} m^2 permeability and 40 m^{-1} extinction coefficient. Selection of better porous media might yield more attractive designs.

Conclusions

Stability criteria for the onset of natural convection in a fluid-saturated porous medium with externally imposed downward flow of the fluid and distributed internal heat generation have been numerically determined from the equations of continuity, motion, and energy. Graphical displays of the results for the onset of natural convection show the relationships between the five pertinent dimensionless parameters of the internal and external Rayleigh numbers, porous medium permeability, extinction coefficient of the fluid-saturated porous medium, and Peclet number based on the externally imposed downward velocity. These stability criteria provide a relationship between the critical internal Rayleigh number and the critical external Rayleigh number.

The results of the present study agree with earlier investigations of the two limiting cases of no downward flow and no internal heat generation, and no downward flow and uniformly distributed internal heat generation.

Appendix: Evaluation of the Coefficients C_m Through F_m and Intergrals I_0 Through I_{11}

Evaluation of the Coefficients C_m Through F_m

The velocity disturbance is given by Eq. (31) as

$$W = \sum_m B_m[\sin(m\pi Z) + C_m \sinh(\beta_1 Z) + D_m \sinh(\beta_2 Z) + E_m \cosh(\beta_1 Z) + F_m \cosh(\beta_2 Z)]$$

Equation (31) is differentiated twice in order to apply the boundary conditions of Eq. (26) with the results that

$$\frac{dW}{dz} = \sum_m B_m[(m\pi)\cos(m\pi Z) + \beta_1 C_m \cosh(\beta_1 Z) + \beta_2 D_m \cosh(\beta_2 Z) + \beta_1 E_m \sinh(\beta_1 Z) + \beta_2 F_m \sinh(\beta_2 Z)] \quad \text{(A1)}$$

$$\frac{d^2W}{dZ^2} = \sum_m B_m[-(m\pi)^2 \sin(m\pi Z) + \beta_1^2 C_m \sinh(\beta_1 Z) + \beta_2^2 D_m \sinh(\beta_2 Z) + \beta_1^2 E_m \cosh(\beta_1 Z) + \beta_2^2 F_m \cosh(\beta_2 Z)] \quad \text{(A2)}$$

Application of the boundary conditions of Eqs. (26a) and (26b) yields the system of equations (A3).

The coefficients C_m through F_m are evaluated by use of Cramer's rule to obtain

$$C_m = \frac{1}{D}[-m\pi \cosh(\beta_1) \sinh(\beta_2)] \quad \text{(A4a)}$$

$$D_m = \frac{1}{D}[m\pi \cosh(\beta_2) \sinh(\beta_1)] \quad \text{(A4b)}$$

$$E_m = \frac{1}{D}[m\pi \sinh(\beta_1) \sinh(\beta_2)] \quad \text{(A4c)}$$

$$F_m = \frac{1}{D}[-m\pi \sinh(\beta_1) \sinh(\beta_2)] \quad \text{(A4d)}$$

where D, the determinant of the system of eqs. (A3), is given by

$$D = \beta_1 \cosh(\beta_1) \sinh(\beta_2) - \beta_2 \cosh(\beta_2) \sinh(\beta_1) \quad \text{(A4e)}$$

Evaluation of the Integrals I_0 Through I_{11}

The integrals I_0 through I_{11} in Eq. (33) are defined as

$$I_0^{nm} = \int_0^1 \exp(-PeZ) \sin(m\pi Z) \sin(n\pi Z)\, dZ \quad \text{(A5a)}$$

$$I_1^n = \int_0^1 \exp(-PeZ) \sinh(\beta_1 Z) \sin(n\pi Z)\, dZ \quad \text{(A5b)}$$

$$I_2^n = \int_0^1 \exp(-PeZ) \sinh(\beta_2 Z) \sin(n\pi Z)\, dZ \quad \text{(A5c)}$$

$$I_3^n = \int_0^1 \exp(-PeZ) \cosh(\beta_1 Z) \sin(n\pi Z)\, dZ \quad \text{(A5d)}$$

$$I_4^n = \int_0^1 \exp(-PeZ) \cosh(\beta_2 Z) \sin(n\pi Z)\, dZ \quad \text{(A5e)}$$

$$I_5^{nm} = \int_0^1 \exp(NZ)\sin(m\pi Z)\sin(n\pi Z)\,\mathrm{d}Z \tag{A5f}$$

$$I_6^n = \int_0^1 \exp(NZ)\sinh(\beta_1 Z)\sin(n\pi Z)\,\mathrm{d}Z \tag{A5g}$$

$$I_7^n = \int_0^1 \exp(NZ)\sinh(\beta_2 Z)\sin(n\pi Z)\,\mathrm{d}Z \tag{A5h}$$

$$I_8^n = \int_0^1 \exp(NZ)\cosh(\beta_1 Z)\sin(n\pi Z)\,\mathrm{d}Z \tag{A5i}$$

$$I_9^n = \int_0^1 \exp(NZ)\cosh(\beta_2 Z)\sin(n\pi Z)\,\mathrm{d}Z \tag{A5j}$$

$$I_{10}^{nm} = \int_0^1 \sin(m\pi Z)\sin(n\pi 2)\,\mathrm{d}Z \tag{A5k}$$

$$I_{11}^{nm} = \int_0^1 \cos(m\pi z)\sin(n\pi Z)\,\mathrm{d}Z \tag{A5l}$$

Evaluation of these integrals yields the following results:

$$I_0^{nm} = \begin{cases} \dfrac{-2\pi^2 mnPe(\exp(-Pe)-1)}{[Pe^2+(m+n)^2\pi^2][Pe^2+(m-n)^2\pi^2]}, & (m-n)\text{ even} \\ \dfrac{2\pi^2 mnPe(\exp(-Pe)+1)}{[Pe^2+(m+n)^2\pi^2][Pe^2+(m-n)^2\pi^2]}, & (m-n)\text{ odd} \end{cases} \tag{A6a}$$

$$I_1^n = \frac{1}{2}(n\pi)\left[\frac{1-\exp(-Pe+\beta_1)\cos(n\pi)}{(-Pe+\beta_1)^2+(n\pi)^2} - \frac{1-\exp(-Pe-\beta_1)\cos(n\pi)}{(Pe+\beta_1)^2+(n\pi)^2}\right] \tag{A6b}$$

$$I_2^n = \frac{1}{2}(n\pi)\left[\frac{1-\exp(-Pe+\beta_2)\cos(n\pi)}{(-Pe+\beta_2)^2+(n\pi)^2} - \frac{1-\exp(-Pe-\beta_2)\cos(n\pi)}{(Pe+\beta_2)^2+(n\pi)^2}\right] \tag{A6c}$$

$$I_3^n = \frac{1}{2}(n\pi)\left[\frac{1-\exp(-Pe+\beta_1)\cos(n\pi)}{(-Pe+\beta_1)^2+(n\pi)^2} + \frac{1-\exp(-Pe-\beta_1)\cos(n\pi)}{(Pe+\beta_1)^2+(n\pi)^2}\right] \tag{A6d}$$

$$I_4^n = \frac{1}{2}(n\pi)\left[\frac{1-\exp(-Pe+\beta_2)\cos(n\pi)}{(-Pe+\beta_2)^2+(n\pi)^2} + \frac{1-\exp(-Pe-\beta_2)\cos(n\pi)}{(Pe+\beta_2)^2+(n\pi)^2}\right] \tag{A6e}$$

$$I_5^{nm} = \begin{cases} \dfrac{2\pi^2 mnN(\exp(-N)-1)}{[N^2+(m+n)^2\pi^2[N^2+(m-n)^2\pi^2]}, & (m-n)\text{ even} \\ \dfrac{-2\pi^2 mnN(\exp(-N)+1)}{[N^2+(m+n)^2\pi^2][N^2+(m-n)^2\pi^2],} & (m-n)\text{ odd} \end{cases} \tag{A6f}$$

$$I_6^n = \frac{1}{2}(n\pi)\left[\frac{1-\exp(N+\beta_1)\cos(n\pi)}{(N+\beta_1)^2+(n\pi)^2} - \frac{1-\exp(N-\beta_1)\cos(n\pi)}{(N-\beta_1)^2+(n\pi)^2}\right] \tag{A6g}$$

$$I_7^n = \frac{1}{2}(n\pi)\left[\frac{1-\exp(N+\beta_2)\cos(n\pi)}{(N+\beta_2)^2+(n\pi)^2} - \frac{1-\exp(N-\beta_2)\cos(n\pi)}{(N-\beta_2)^2+(n\pi)^2}\right] \tag{A6h}$$

$$I_8^n = \frac{1}{2}(n\pi)\left[\frac{1-\exp(N+\beta_1)\cos(n\pi)}{(N+\beta_1)^2+(n\pi)^2} + \frac{1-\exp(N-\beta_1)\cos(n\pi)}{(N-\beta_1)^2+(n\pi)^2}\right] \tag{A6i}$$

$$I_9^n = \frac{1}{2}(n\pi)\left[\frac{1-\exp(N+\beta_2)\cos(n\pi)}{(N+\beta_2)^2+(n\pi)^2} + \frac{1-\exp(N-\beta_2)\cos(n\pi)}{(N-\beta_2)^2+(n\pi)^2}\right] \tag{A6j}$$

$$I_{10}^{nm} = \begin{cases} 1/2 & n=m \\ 0 & n\neq m \end{cases} \tag{A6k}$$

$$I_{11}^{nm} = \begin{cases} 0, & (n-m)\text{ even} \\ \dfrac{2n}{\pi(n-m)(n+m)}, & (n-m)\text{ odd} \end{cases} \tag{A6l}$$

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